A Simple Investigation into the Limiting Space of an Edge Replacement System

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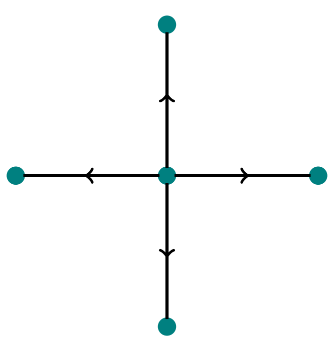
Introduction

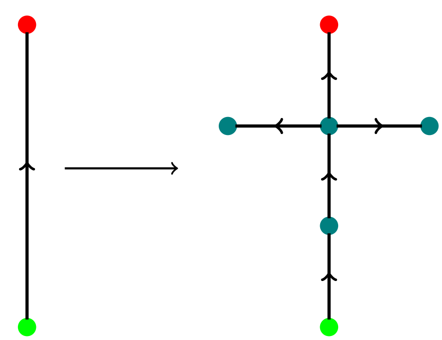
This is a small project on determining the limiting space of an edge replacement system. Information on edge replacement systems can be found in the paper "*Rearrangement Groups of Fractals*" by James Belk and Bradley Forrest. Alternatively, there will be a project on edge replacement systems posted on my website in the near future.

The Edge Replacement System

The edge replacement of focus is $\mathcal{R} = (G_0, e \to R)$, where G_0 and R are given by



G_0



$e \to R$

The red and green vertices are the initial and terminating vertices respectively. We will get the limiting space of $\mathcal{R}$.

First, we will show that $\mathcal{R}$ is expanding, which implies that the gluing relation on the symbol space Ω of $\mathcal{R}$ is an equivalence relation. There are three conditions that need to be met, and it is clear that $\mathcal{R}$ satisfies them:

1. G_0 and R do not have isolated vertices.
2. The initial and terminating vertex of R are not adjacent.
3. R has at least three vertices and two edges.

To obtain the limiting space of $\mathcal{R}$, we need to know which sequences in Ω are equivlalent. We will do this by looking at the first few elements of the full expansion graph, which has been drawn in Figure 2. Two elements $\epsilon_0\epsilon_1\epsilon_2\ldots,\ \epsilon'_0\epsilon'_1\epsilon'_2\ldots \in \Omega$ are equivalent if all $n \geq 0$, the edges $\epsilon_0\epsilon_1\epsilon_2\ldots\epsilon_n$ and $\epsilon'_0\epsilon'_1\epsilon'_2\ldots\epsilon'_n$ share a vertex in G_n. If we look at the centre of each graph in the full expansion sequence, then we see that $T\bar{0} \sim L\bar{0} \sim R\bar{0} \sim B\bar{0}$. We create new 'centres' in each successive graph of the full expansion sequence, hence we can see that

$$\epsilon_0\ldots\epsilon_n 1\bar{0} \sim \epsilon_0\ldots\epsilon_n 2\bar{0} \sim \epsilon_0\ldots\epsilon_n 3\bar{0} \sim \epsilon_0\ldots\epsilon_n 4\bar{0}.^*$$

Finally, if we look at the regions of G_n where two edges point to each other, then we see that

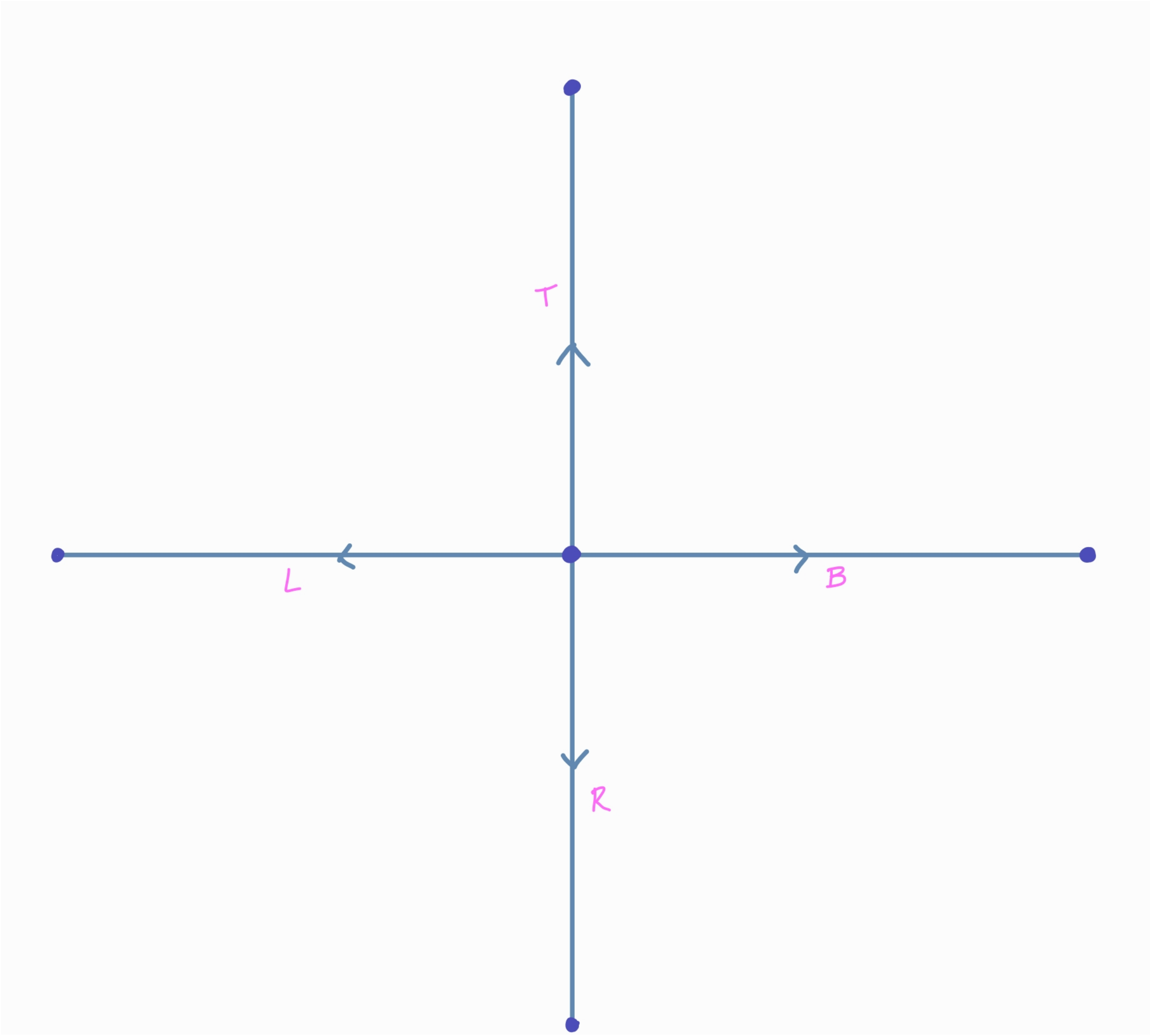
$$\epsilon_0\ldots\epsilon_n 0\bar{3} \sim \epsilon_0\ldots\epsilon_n 1\bar{3}.$$

The limiting space of $\mathcal{R}$ is $\Omega/\sim$. This space is homeomorphic to the Vicsek fractal.

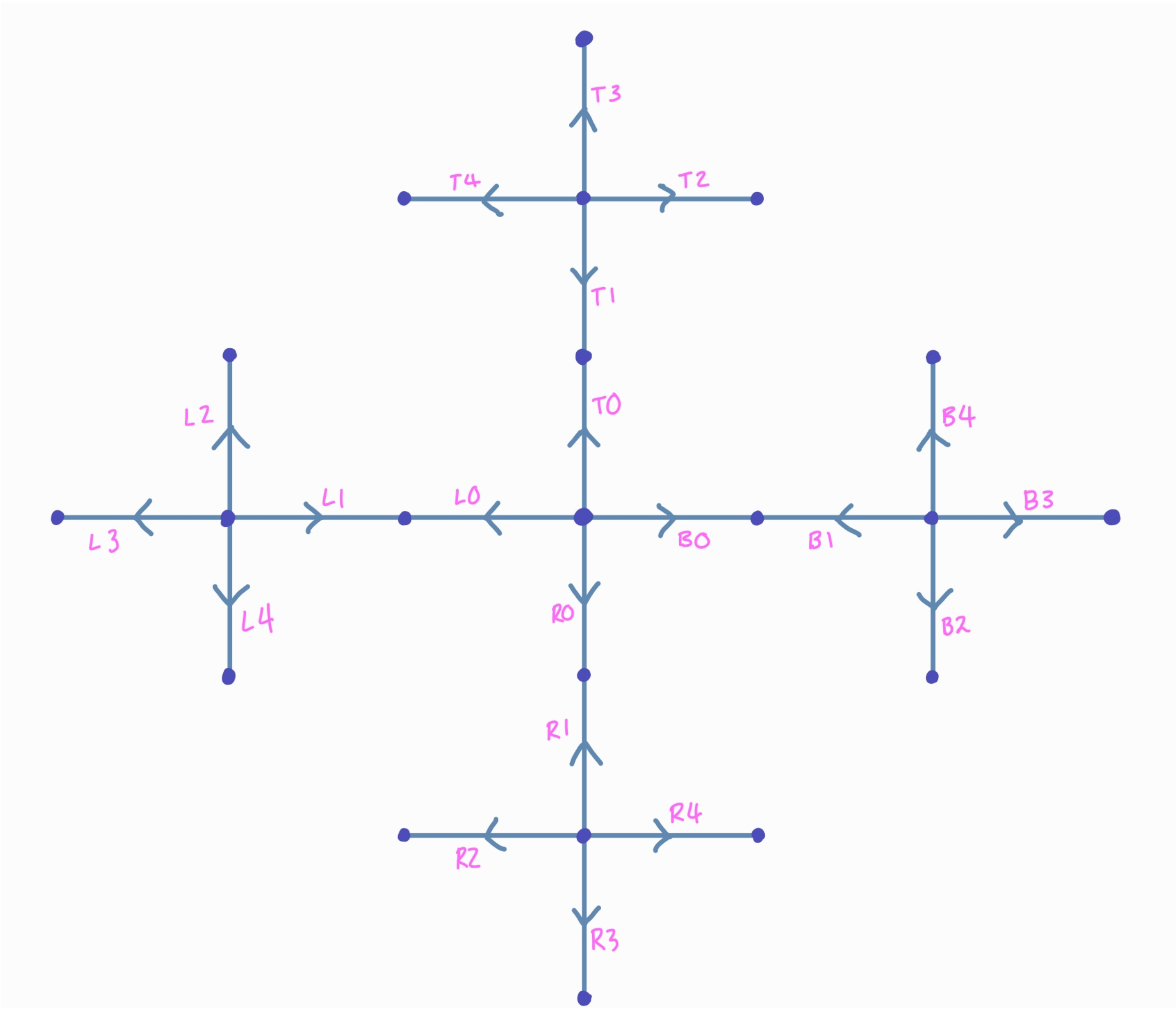
*Where $\epsilon_0 \in E(G_0)$ and $\epsilon_i \in E(R)$ for $i > 0$.

Figure 2: First three elements of the full expansion graph.

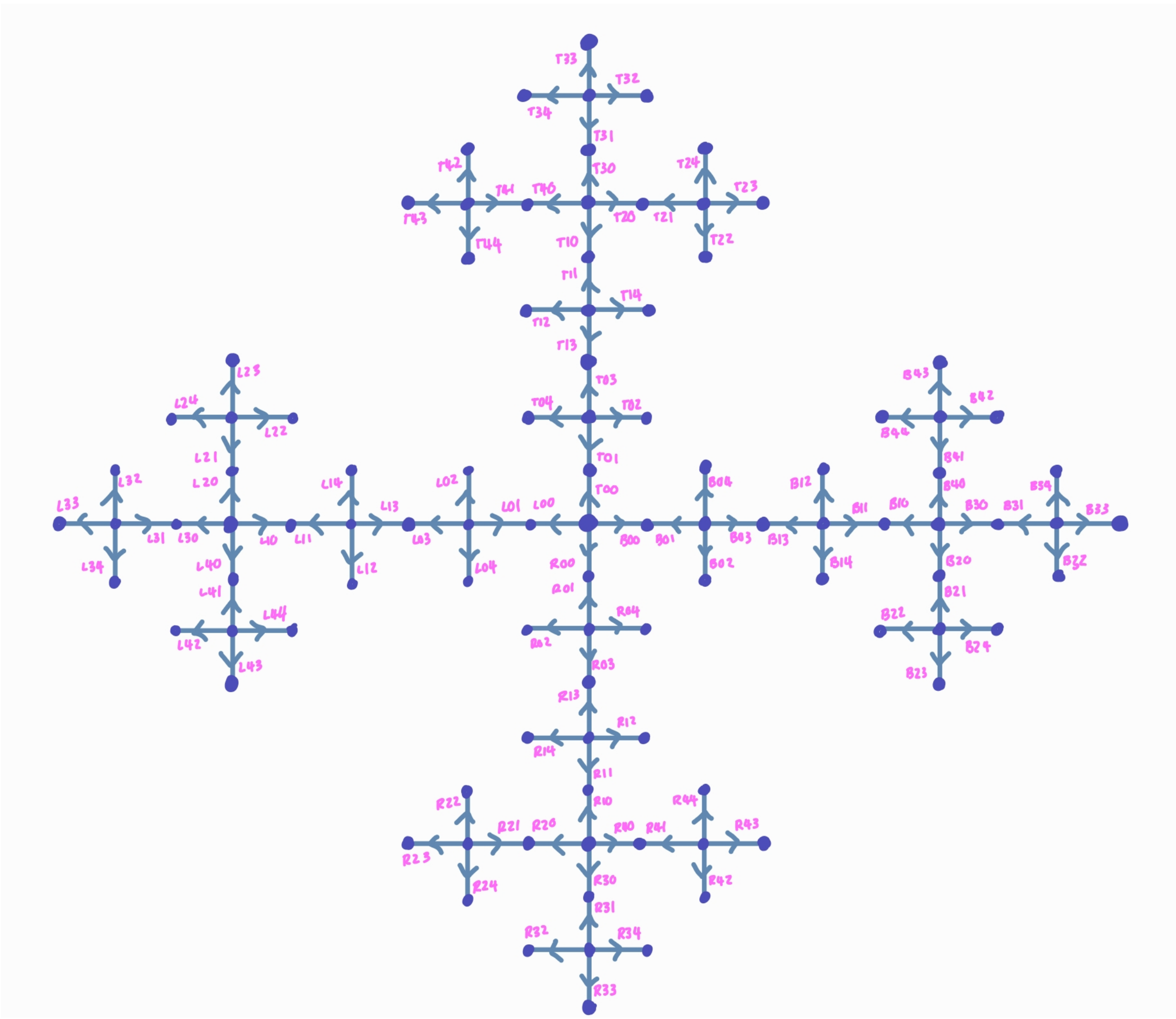
(a) G_0



(b) G_1



(c) G_2



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